THERMOCAPILLARY CELL IN A HORIZONTAL MOVING LIQUID LAYER WITH HEATING FROM ABOVE

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The problem of thermocapillary convection that develops in a thin stationary liquid layer, heated locally from above, has been considered in [1] in the constant layer thickness approximation. The exact solution of equations of capillary convection that had been obtained in [1] for large Marangoni numbers ($M \gg 1$) made it possible to determine the convection rate and the dimensions of the convective cell. The solution given in [1], which holds for the Prandtl numbers $Pr \geq 1$, has been generalized in [2] to encompass the case of liquid metals ($Pr < 1$). The problem from [1] was considered in its rigorous statement in [3] with an allowance for the curvature of the free liquid surface. This made it possible to determine the effect of gravity and surface deformation on the cell structure and define the applicability scope of the simple analytical solution [1, 2]. On the basis of the results obtained in [1-3], a dyanmic thermocapillary model of the vacuum-arc cathode spot has been proposed in [4]. This model relates the high speed of a cathode spot moving over the electrode surface to the high rate of convective heat transfer. It is interesting to examine the problem of steady-state thermocapillary convection in a thin, horizontal moving liquid layer with local heating from above.

As in [1], we shall limit our considerations to a two-dimensional model for $M \gg 1$. If the liquid is immobile, two symmetric thermocapillary cells are formed on either side of the heating line. The liquid motion initiated by the capillary force encompasses a finite length, the cell length $l \simeq hM^{1/2} \gg h$, where h is the layer thickness. The maximum flow velocity develops at the surface and is equal to

$$v_c \simeq (\alpha' \Delta T \chi / h\eta)^{1/2},$$

where $\alpha' = -d\alpha/dT$, α is the surface tension coefficient, $\Delta T = T_0 - T_1$ is the temperature drop at the initial cross section, χ is the temperature diffusivity, and η is the dynamic viscosity. According to [1], the vortex intensity is characterized by the following volumetric (per unit length) discharge of the circulating liquid:

$$Q_c \simeq 0.15(h\alpha' \Delta T \chi/\eta)^{1/2} = 0.15\chi M^{1/2}.$$

The question arises whether the cellular convection structure persists if the liquid layer heated from above moves and is characterized by the discharge $Q \neq 0$. The cell on the side of incoming flow may persist, at least if $Q \leq Q_c$. Its structure will change, but one can expect that its length will in this case diminish slightly, so that, as before, we can use the boundary layer approximation to describe the convection. The cell on the other side of the heating line may, generally, be destroyed since liquid particles from the heating location may drift away to infinity. We shall solve the problem in the approximation of constant layer thickness. This is justified if [3]

$$(\alpha' \Delta T/\rho g h^2) \leqslant 1,$$

where ρ is the liquid density, and g is the acceleration due to gravity. Assume that the liquid layer is bounded by its free surface, $y = 0$, and the vessel bottom, $y = -h$. We place the x axis in opposition to the temperature gradient along the layer. For a liquid with the Prandtl number $Pr \geq 1$, the initial system of equations and boundary conditions is given by [1]

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0, \quad \frac{\partial p}{\partial x} = \eta \frac{\partial^2 v_x}{\partial y^2}, \quad \frac{\partial p}{\partial y} = 0, \quad v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \chi \frac{\partial^2 T}{\partial y^2}; \tag{1}$$

$$v_x(x, -h) = v_y(x, -h) = 0, \quad T(x, -h) = T_1, \quad \left.\frac{\partial T}{\partial y}\right|_{y=0} = 0,$$
$$\eta \left.\frac{\partial v_x}{\partial y}\right|_{y=0} = \frac{d\alpha}{dx} = -\alpha' \left.\frac{\partial T}{\partial x}\right|_{y=0}. \tag{2}$$

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The latter equation in (2) expresses the continuity of the stress tensor at the free surface. The kinematic condition at the free surface is in this case reduced to the requirement for $v_y(x, 0) = 0$; as will be shown in the solution, this condition is satisfied automatically. With an allowance for (2), we have the following integral on the basis of the continuity equation:

$$\int_{-h}^{0} v_x dy = Q = \text{const.} \tag{3}$$

For the cell on the side of oncoming flow, $Q < 0$, while $Q > 0$ for the flow on the opposite side from the heating location. By integrating the equations of motion and continuity and considering (3), we find the expressions for the velocity

$$v_x = -\frac{3\alpha'}{4h\eta}\left(\frac{\partial T}{\partial x}\right)_{y=0} G'(y) - \frac{3Q}{2h^3}(y^2 - h^2), \quad v_y = \frac{3\alpha'}{4h\eta}\left(\frac{\partial^2 T}{\partial x^2}\right)_{y=0} G(y), \tag{4}$$

where $G = (1/3)y(y + h)^2$. We introduce the dimensionless variables $\theta = (T - T_1)/\Delta T$, $x' = x/h$, and $y' = y/h$. By substituting (4) in the equation of heat transport in the liquid, we obtain (the primes will be subsequently omitted)

$$\left[-G'\left(\frac{\partial\theta}{\partial x}\right)_0 + q(1 - y^2)\right]\frac{\partial\theta}{\partial x} + G\left(\frac{\partial^2\theta}{\partial x^2}\right)_0 \frac{\partial\theta}{\partial y} = \frac{4}{3\mathrm{M}}\frac{\partial^2\theta}{\partial y^2}, \tag{5}$$

where $q = 2Q\eta/\alpha'\Delta Th$. For $q = 0$, Eq. (5) has an exact particular solution [1] describing the thermocapillary cell, whose length is determined from the one-dimensional linear spectral problem. For $q \neq 0$, the variables cannot be separated, and the exact solution in explicit form cannot be found. In order to obtain an approximate solution of (5), we expand θ in a series with respect to y and reject terms with $\sim y^3$ and higher-power terms. Then, having satisfied (2), we arrive at the parabolic approximation of the temperature profile across the layer:

$$\theta \simeq X(x)(1 - y^2) = X\Phi. \tag{6}$$

As will be shown below, approximation (6) is sufficient for an approximate analysis of the cell structure on the oncoming flow side. After substituting (6) in (5), we obtain a differential equation for determining X from the requirement for orthogonality of the residue of the equation with respect to Φ. The above approach, which is similar to the procedure used in deriving Galerkin's equations, makes it possible to find the solution as a function of x in explicit form. Instead of Φ in (6), we can use a polynomial of a higher power or, for instance, the exact eigenfunction of the nonperturbed problem. The numerical coefficients would change in the equation for X, but neither its structure nor, which is most important, its solution at the physical level of rigorousness would change. However, this would hardly be worthwhile since the solution can be represented in the form (6) only with an accuracy to terms with $\sim y^2$ inclusively.

Omitting the mathematical operations, we write the equation for X:

$$XX'' + 2X'^2 - \mu X' - AX = 0, \tag{7}$$

where

$$\mu \simeq 28.8q, \quad A = 10h^2/l_0^2, \quad l_0 \simeq 0.3h\sqrt{\mathrm{M}}. \tag{7'}$$

It is necessary to find the bounded (with finite first and second derivatives) solution of (7) that would satisfy the condition $X(0) = 1$. The parameter μ appears in (7) in a regular way. Assuming that it is small, we use the perturbation theory:

$$X = X_0 + \mu X_1 + \mu^2 X_2 + \ldots$$

As will be seen below, the actual parameter of the expansion is the Q/Q_c ratio. This parameter is small for oncoming flow velocities $v < v_c$. The solution of the equation for X_0 that satisfies the stipulated conditions is given by

$$X_0 = (1 \pm x/l_0)^2. \tag{8}$$

In the case where the temperature decreases with an increase in x, the solution (8) with the minus sign should be used. The form of this expression coincides with that of the exact solution [1]. According to (8), the quantity l_0 has the sense of the characteristic linear dimension of the cell, or, more accurately, it is equal to the cell length for $\mu = 0$. (It should be noted that the numerical coefficient 0.3 appearing in the l_0 definition (7') is in fairly good agreement with the coefficient 0.52 obtained by more accurate calculations [1]. This betokens the advisability of using the chosen approximation (6). In calculating the following terms of the expansion, it is convenient to use a new independent variable, $\xi = 1 - x/l_0$. Then, the equation for X_1 is written in the following manner:

$$\xi^2 X_1'' + 8\xi X_1' - 8X_1 + \frac{2l_0}{h}\xi = 0, \quad X_1(1) = 0.$$

Its solution having a physical meaning is given by

$$X_1 = -\frac{2l_0}{9h}\xi \ln|\xi|. \tag{9}$$

Omitting the mathematical operations, we provide the equation for X_1 and its solution:

$$\begin{gathered} \xi^2 X_2'' + 8\xi X_2' - 8X_2 + \frac{2l_0^2}{(9h)^2}(\ln|\xi| + 4\ln^2|\xi| - 5) = 0, \\ X_2 = \frac{3}{4}\left(\frac{l_0}{9h}\right)^2(1-\xi) + \left(\frac{l_0}{9h}\right)^2(\ln^2|\xi| + \ln \xi^2). \end{gathered} \tag{10}$$

According to (9) and (10), as was mentioned above, the actual parameter of the expansion of the solution is the quantity

$$\varepsilon = \mu l_0/9h \simeq 0.29Q/Q_c.$$

For simplicity of notation, we shall give the results of the solution analysis in an approximation which is linear with respect to ε. The second approximation involves quantitative corrections without altering the character of the distributions. In order to find the roots (determine the cell length), the solution should be written in the following form:

$$X = (\xi - \varepsilon \ln|\xi|)^2. \tag{11}$$

With an accuracy to the value of $\sim\varepsilon^2$, this expression coincides with the sum of (8) and (9). It is evident from (11) that, for $\varepsilon < 0$, there is a double root, $\xi = \xi_1 (0 < \xi_1 < 1)$. In the $\xi_1 \le \xi \le 1$ interval, the terms of the expansion of X and their derivatives are smooth functions. In other words, solution (11) describes in the above interval the thermocapillary cell formed in the liquid on the oncoming flow side. If $\varepsilon > 0$, (11) has the root $-1 < \xi_2 < 0$. In this case, the solution in the $\xi_2 < \xi < 1$ interval is singular and has no physical meaning, i.e., more stringent analysis is necessary for describing the convection pattern down stream from the heat source. We shall limit our analysis to the first case ($\varepsilon < 0$).

At the point $\xi = \xi_1$, the perturbation of the liquid temperature due to the source vanishes, i.e., the quantity $\Delta\xi = 1 - \xi_1$ represents the dimensionless length (in l_0 units) of the thermocapillary cell. The dependence of $\Delta\xi$ on ε is given in Table 1, which indicates that the rate of reduction in the cell length diminishes with an increase in $|\varepsilon|$. Substituting (11) in (4), we find the final expression for the velocity field:

$$\begin{aligned} v_x/v_m &= (\xi + |\varepsilon|\ln\xi)\left(1 + \frac{|\varepsilon|}{\xi}\right)(1+y)(1+3y) - 0.47|\varepsilon|(1-y^2), \\ v_y/v_m &= \frac{h}{l_0}\left(1 + \frac{|\varepsilon|}{\xi}\right) y(1+y)^2. \end{aligned} \tag{12}$$

where $v_m = 1.66v_c$ is the maximum velocity at the free surface at the initial section.

It is evident from (12), that the liquid motion consists of a superposition of the thermocapillary vortex on Poiseuille flow. Let us examine the following fact. For $\varepsilon = 0$, at the cell boundary $\xi_1 = 0$, the horizontal velocity component vanishes together with X and X', so that the entire thermal flux from the source vanishes as well. According to (12), for $\varepsilon \neq 0$, v_x vanishes for $\xi = \xi_0 > \xi_1$. The quantities $\xi_0 - \xi_1$ and $X(\xi_0)$ are small. For instance, if $\varepsilon = -0.2$, we have $\xi_0 \approx 0.298$, $\xi_1 \approx 0.265$, $X(\xi_0) \approx 0.003$, and $\xi_0 - \xi_1 \to 0$ for $|\varepsilon| \to 0$. It is inadvisable to draw conclusions on the basis of the difference between ξ_0 and ξ_1

TABLE 1

$\lvert\varepsilon\rvert$	0	0,01	0,05	0,1	0,2	0,3
$\Delta\xi$	1	0,966	0,889	0,825	0,735	0,669

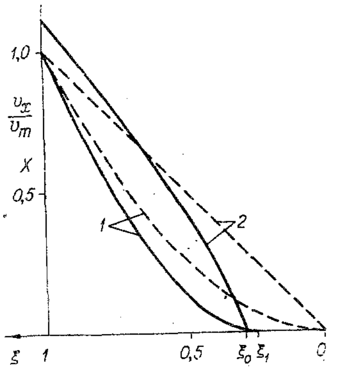

Fig. 1

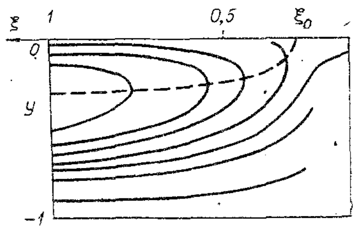

Fig. 2

or to examine details of the convection pattern near the cell boundary within the framework of this approach. In order to clarify this, we shall discuss the admissibility of using the boundary layer approximation for describing cellular convection. According to (12), v_y is the value of $\sim(h/l_0)v_m$. The condition $v_y \ll v_x$ can be violated only in the vicinity of the line where $v_x = 0$. It is evident from (12) that $v_y \sim v_x$ in a strip with the width $\Delta y \sim h/l_0$ around the line in question. In this region of low (relative to v_m) velocities, we must use the Navier–Stokes equations. Since this region is small ($\sim M^{1/2}$) in comparison with the cell area, one could expect that the error caused by the use of inexact equations in this region would be small in describing the cell properties as a whole. However, this is generally not so if one considers the structure of the transitional region between the cell and the unperturbed flow. The limiting value of the coordinate where $v_x = 0$ is the point $\xi = \xi_0$. Therefore, it only makes sense to consider the obtained solution for $\xi \geq \xi_0$.

For the assigned temperature drop, the reduction in the cell length for $\varepsilon \neq 0$ causes an increase in the gradient of the surface tension coefficient, and, consequently, the shearing stress at the free surface. Therefore, the convection rate also increases. This is illustrated in Fig. 1, where curves 1 represent the temperature distribution (the dashed curve pertains to $\varepsilon = 0$, while the solid curve pertains to $\varepsilon = -0.2$), while curves 2 represent the corresponding v_x/v_m distributions at the free surface. For a moving liquid, the drop in the temperature and the convection rate with increasing distance from the initial section is steeper than the drop governed by the parabolic and linear laws, respectively, for $\varepsilon = 0$. Figure 2 shows the streamlines calculated on the basis of (12) for $\varepsilon = -0.2$ in a (ξ, y) coordinate system. The line at which $v_x = 0$ is shown by the dashed curve. At the points of intersection with this curve, the streamlines have a vertical tangent (at $\varepsilon = 0$, the flow reversal line $y = -1/3$ has a lower position). The motion region can be divided into two zones by a streamline issuing from the point (ξ_0, 0) in the direction perpendicular to the axis of abscissas (the boundary line is not shown, and the diagram provides only the nearby streamlines). The upper zone constitutes a " floating" thermocapillary cell with the total liquid flow equal to zero at any section ξ = const. The lower zone represents a streamtube with the assigned discharge Q, in which everywhere $v_x < 0$. Roughly speaking, the oncoming flow dives, as it were, under the vortex cell formed by the capillary force, compressing it relatively heavily in the longitudinal, and slightly in the transverse, directions. It is evident from Fig. 2 that, for $\xi \geq \xi_0$, the solution describes satisfactorily the convection pattern. It is interesting to note that the vortex intensity does not change, as measured by the magnitudes of circulating liquid flow. The depicted convection pattern on the oncoming flow side suggests the possibility of existence of a thermocapillary cell on the other side of the heating location under certain conditions.

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LITERATURE CITED

1. Yu. V. Sanochkin, "Thermocapillary convection in a thin liquid layer heated locally from above," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1983).
2. Yu. V. Sanochkin, "Steady-state thermocapillary motion in a horizontal liquid metal layer heated locally from above," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 6 (1984).
3. S. I. Vyboronov and Yu. V. Sanochkin, "Thermocapillary cell in a heavy liquid layer from above," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 1 (1985).
4. A. V. Zharinov and Yu. V. Sanochkin, "Possible explanation of the mechanism of cathode spot motion," Pis'ma Zh. Eksp. Teor. Fiz., 36, No. 5 (1982).

EFFECT OF SPATIAL MODULATION OF THE TEMPERATURE DISTRIBUTION ON THE STABILITY OF TWO-DIMENSIONAL STEADY FLOW IN A HORIZONTAL LAYER OF A TWO-COMPONENT LIQUID

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We study the stability of two-dimensional steady flow in a horizontal layer of viscous heat-conducting liquid containing an admixture. For constant temperatures of the boundaries of the layer the convection equations admit a steady-state solution (mechanical equilibrium) which is stable if the temperature gradient is not too large. Under spatial modulation of the temperature distribution the liquid cannot be in equilibrium, and a spatially periodic convective regime is established in it for arbitrarily small temperature gradients [1, 2]. The purpose of the present article is to find the critical values of the temperature gradient for which this primary regime becomes unstable and a secondary regime develops in the liquid. A similar problem was solved in [2] for a homogeneous liquid when both boundaries of the layer are free surfaces.

1. Formulation of the Problem. Suppose a viscous heat-conducting liquid containing an admixture fills an infinite plane horizontal layer of thickness h. The lower boundary of the layer is a solid surface whose temperature is modulated by small-amplitude perturbations which are periodic along the layer. The free upper surface of the layer is not deformed (taking account of the deformability is important only for thin layers of liquid and in weak gravitational fields [3]), and it is free of tangential stresses. The atmosphere above the layer is a stationary gas having a quasistationary temperature distribution. The heat flux Q along the vertical in the atmosphere far from the free surface is assumed given (for heating from below $Q > 0$). We assume that the temperature and the normal component of the heat flux are continuous through the free surface. There is no flow of the admixture through the boundaries of the layer. The liquid as a whole cannot be displaced parallel to the bottom. The amount of admixture in the liquid is specified.

The problem of determining the velocity $\mathbf{v} = \{v_x, v_y, v_z\}$, the pressure Π, the temperature T of the liquid, the temperature Θ of the atmosphere, and the concentration S of the admixture, reduced to dimensionless form and written in the Boussinesq approximation, has the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v} = -\nabla\Pi + \Delta\mathbf{v} + \mathbf{e}(GT - G_s S),$$
$$\frac{\partial T}{\partial t} + (\mathbf{v}, \nabla)T = \frac{1}{\mathrm{Pr}}\Delta T, \quad \operatorname{div}\mathbf{v} = 0, \quad \Delta\Theta = 0, \tag{1.1}$$
$$\frac{\partial S}{\partial t} + (\mathbf{v}, \nabla)S = \frac{1}{\mathrm{Pr}_d}\operatorname{div}(\nabla S + \xi S\nabla T),$$

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